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Sums of Valences in Bigraphs

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Let S be a set of $2n$ points in general position in the plane; then any line lying on two of these points and having $n - 1$ of the points of S on each side of it is called a bisector of S. We show that if p and q are distinct points of S having ν_p and ν_q bisectors on them, respectively, then $\nu_p + \nu_q \leqslant 2n$. The development leading to this result is based upon a fundamental property concerning the relative positions of the bisectors on a point. The sum $\nu_p + \nu_q + \nu_r$ is also discussed.

INTRODUCTION

Since bigraphs and bisectors of point sets apparently have no prior literature, it will be necessary to begin this paper by giving several key definitions.

Let S be a set of $2n$ points in general position in the plane; then any line lying on two of these points, p and q, and having $n - 1$ of the points of S on each side of it is called the *set bisector* (or *bisector* for short) determined by p and q and is written pq. If p and q determine a bisector, p and q are said to be *adjacent*. It will be our usual convention in constructing the graphic representation of the bisector determined by p and q to draw only that segment of pq which lies between p and q. The figure consisting of the set of all possible such segments of bisectors pq determined by the points of S is called the *bigraph* determined by S. The number of bisectors incident on a point p in the bigraph is called the *valence* of p, written ν_p. It is obvious for the upper bound and simple to show for the lower that $1 \leqslant \nu_p \leqslant 2n - 1$. We will show that if p and q are distinct points then $\nu_p + \nu_q \leqslant 2n$. That this bound is best is shown by the bigraph on the set S with $2n - 1$ of its points equally spaced on a circle and its remaining point at the center of the circle. At the same time we show $\nu_p + \nu_q \leqslant 2n - 2$ if p and q are not adjacent. These results lead to an upper bound for the sum of the valences of three distinct points and we show by constructions that this bound is obtained.

Figure 1 shows the only possible bigraphs on 2, 4, and 6 points, assuming equivalence of the figures under all transformations which leave the topologic character of the bigraphs unaffected.

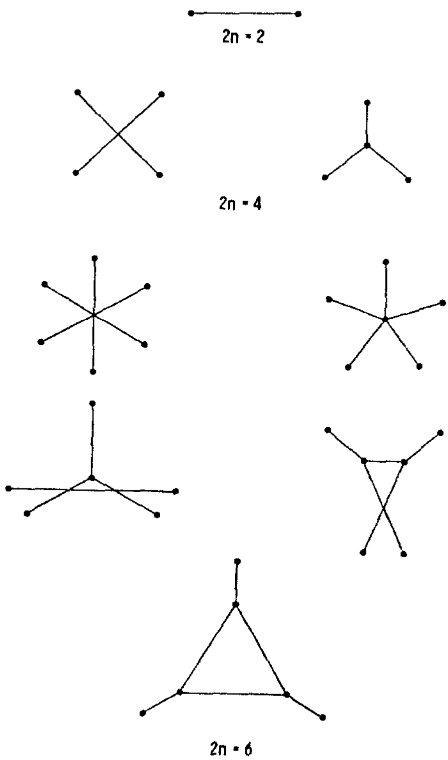
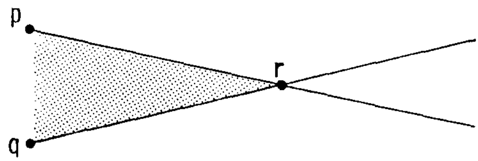


FIGURE 1

PARITY IN BIGRAPHS

If p, q, and r are three points in general position in the plane then the two lines, one on p and r, the other on q and r, determine four open regions in the plane. The convex open region bounded by the half lines from r and on p and q is called the (*positive*) *cone* of prq and written prq while that open region not adjacent to prq is called the *negative cone* of prq and written $-prq$.



THEOREM 1. *If p, q, and r are points of S with r adjacent to both p and q then r is adjacent to a point of S in $-prq$.*

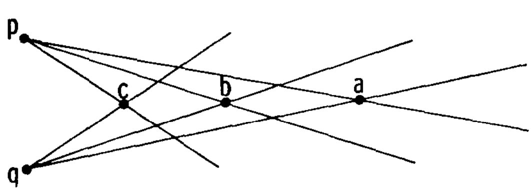
Proof. We may assume p, r, and q form a clockwise triple as illustrated and that r is not adjacent to any point of S in prq. Rotating the bisector pr counterclockwise through a sufficiently small angle about r as center causes there to be n points on the side of the rotated line opposite q and $n - 1$ points on the other side. On the other hand, when the bisector pr is rotated sufficiently close to q there must be $n - 1$ points on the side of the rotated line opposite q and n points on the other side. Since the rotating line can cross only one point at a time as it rotates on r (the points are in general position), at some position there must have been $n - 1$ points on both sides of the line, i.e., r is adjacent to a point in $-prq$. ∎

The following corollary is a direct consequence of the remark made earlier that for every point p of S, $1 \leqslant \nu_p$ and of the preceding theorem.

COROLLARY. *In a bigraph every point of the determining set has an odd valence.*

PARTIAL ORDERING OF S

Given an arbitrary pair of distinct points p and q in S we define a partial order on the remaining points by the rule that $a > b$ if and only if a lies in $-pbq$. That this defines a partial ordering is easy to see. First $a > b$ and $b > a$ implies $a = b$, since $a > b$ says that $-paq$ is entirely contained in $-pbq$, while $b > a$ implies containment in the opposite direction. Therefore the cones must be identical and hence $a = b$. A similar elementary geometric argument based on inclusion establishes transitivity:



If $a > b$ and a is adjacent to b, then the bisector ab intersects the line segment between p and q. Such a bisector is called an *internal bisector* on a (with respect to p and q).

LEMMA. *If $a > b_1$ and $a > b_2$ and a is adjacent to b_1 and b_2 then there exists a point c adjacent to a such that $c > a$.*

Proof. Given that a is adjacent to b_1 and b_2 Theorem 1 has shown that there exists a point c adjacent to a such that ac is an internal bisector on c with respect to b_1 and b_2. However, $a > b_1$ and $a > b_2$ with respect to p and q so that the line segment between b_1 and b_2 is completely in triangle paq; hence ac is also an internal bisector on c with respect to p and q so that $c > a$. ∎

MAXIMUM FOR THE SUM $\nu_p + \nu_q$

Given an arbitrary pair of distinct points p and q in S each of the remaining points occurs in just one of the three sets:

I. points adjacent to both p and q,

II. points adjacent to p or q but not both,

III. points adjacent to neither p nor q.

THEOREM 2. *If $a > b_i$, $i = 1,\ldots, k$, and a is adjacent to each b_i, then there exist bisectors ac_j, $j = 1,\ldots, m$ where $c_j > a$ and*

1. $m = k + 1$, if $a \in \mathrm{I}$,
2. $m = k$, *if* $a \in \mathrm{II}$,
3. $m = k - 1$, *if* $a \in \mathrm{III}$.

Proof. By the lemma all of the ab_i intersect the line segment on p and q internally and since $a > b_i$ for all i all of the b_i are in paq. If $a \in \mathrm{II}$ then either ap or aq exists and if $a \in \mathrm{I}$ they both exist. By Theorem 1 therefore m bisectors ac_j must exist with c_j in $-paq$, i.e., $c_j > a$. ∎

We define a sequence of points $a_1, a_2, \ldots, a_r$ of S to be an *adjacency chain* if and only if $a_i < a_{i+1}$ and a_1 and a_j are adjacent for $i = 1, 2,\ldots, r - 1$. By the definition of internal bisectors it is obvious that all of the a_ia_{i+1} are internal bisectors with respect to p and q. Using this concept it is now possible to state and prove the following basic counting theorem.

THEOREM 3. *If class I contains m points then there are at least m points in class III.*

Proof. For each $a \in \mathrm{I}$ form an adjacency chain of maximal length; $a = a_1, a_2, \ldots, a_r = a'$ by selecting a_{i+1} so that $a_i a_{i+1}$ is the clockwisemost available bisector with a_{i+1} in $-pa_iq$. Mark as unavailable for the formation of other adjacency chains those bisectors used in forming this chain. First we note that two such maximal chains can never terminate on the same point. By Theorem 2, if k chains are incident on a point a of class I, $k + 1$ bisectors ac_j with each c_j in $-paq$ exist so that all k chains can continue as well as providing a first step for the chain which must originate on a since a is in class I. If a is in class II then there are the same number of ac_j with c_j in $-paq$ as there are incident chains so that all chains continue on through a, while if a is in class III all but possibly one chain continue and this last chain may terminate on a. Second, we note that a' cannot be adjacent to p or q since by the above discussion it must be in class III. Therefore every maximal length chain terminates on a distinct element of class III as was to be shown. ∎

COROLLARY. *Let p and q be arbitrary distinct points in a bigraph on $2n$ points then $\nu_p + \nu_q \leqslant 2n - 2$ if p is not adjacent to q and $\nu_p + \nu_q \leqslant 2n$ if p is adjacent to q.*

Proof. Given p and q the remaining $2n - 2$ points can contribute a maximum of $2n - 2$ to $\nu_p + \nu_q$ if and only if every point of class III is the end point of a maximal length chain originating on a point in class I. ∎

It easily follows from the corollary that a bigraph has at most $\frac{1}{2}\binom{n}{2}$ bisectors if $n > 1$; Lovász has shown [1] that a bigraph has at most $2n\sqrt{2n}$ bisectors.

MAXIMUM FOR THE SUM $\nu_p + \nu_q + \nu_r$

THEOREM 5. *Let p, q, and r be distinct points in a bigraph on $2n$ points. Then*

$$\nu_p + \nu_q + \nu_r \leqslant 3n - 1$$

if n is even, and

$$\nu_p + \nu_q + \nu_r \leqslant 3n$$

if n is odd. Furthermore these bounds are the best possible in the sense that for every n that there exists a bigraph on $2n$ points for which the sum of three valences achieves the indicated bound.

Proof. Since by the corollary to Theorem 3,

$$\max_{\substack{p,q\in S\\ p\neq q}} (\nu_p + \nu_q) \leqslant 2n,$$

it follows that

$$\max_{\substack{p,q,r\in S\\ p\neq q\neq r\neq p}} (\nu_p + \nu_q + \nu_r) \leqslant 3n$$

in any case. If n is even so that $3n$ is also even the upper bound $3n$ can be replaced by $3n - 1$ since the individual valences must be odd by the corollary to Theorem 1. We prove that these bounds can be achieved by construction. The basic building block for this construction is the bigraph on six points which we call a *central symmetric star bigraph* of *order* 0. Consider a circular locus inside of the central triangular region of this bigraph as shown in Figure 2. If k points are situated on this circle in

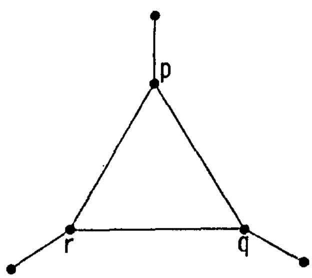


FIGURE 2

general position with respect to the original six points and are adjacent to p, q, and r, then by Theorem 2 there must be $k + 1$ bisectors on each of p, q, and r and having their other determining points in $-qpr$, $-pqr$, and $-qrp$, respectively. This causes there to be $2k + 3$ bisectors incident on each of the points p, q, and r, 3 bisectors on each of the k points in the center, and a single bisector on each of the $3k + 3$ points in the negative cones. The resulting configuration has $4k + 6$ points and $6k + 6$ bisectors; however, the important point for the present discussion is that

$$\nu_p + \nu_q + \nu_r = 3(2k + 3) = 3n,$$

where $n = 2k + 3$ is necessarily odd.

Figure 3 shows the central symmetric star bigraphs of orders 0, 1, 2, and 3.

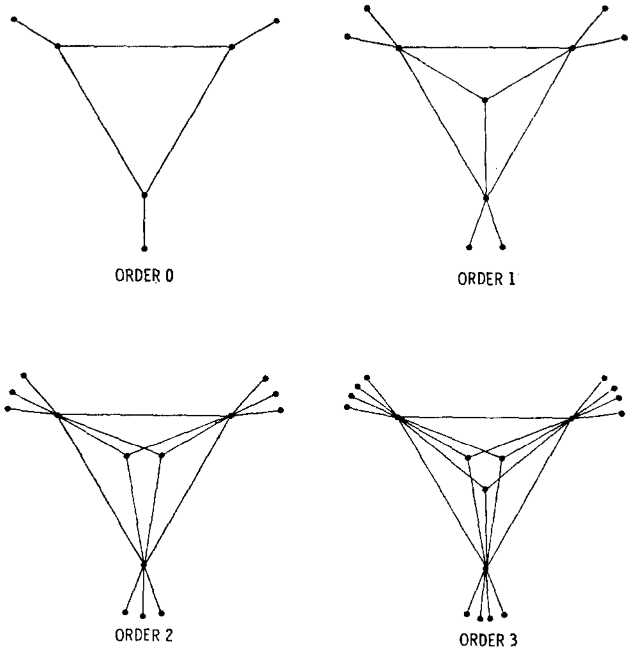


FIG. 3. Central symmetric star bigraphs.

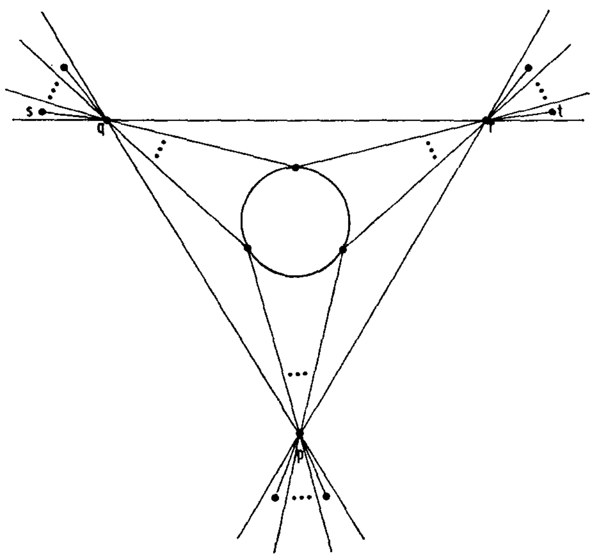


FIG. 4. Order k central star bigraph.

The construction for the case in which n is even is based on removing a pair of points (and the three associated bisectors) from the central symmetric star bigraph on $2(n + 1)$ points. Consider the k-th order central symmetric star bigraph as shown schematically in Figure 4. If we remove points s and t and hence bisectors qs, qr, and rt it is easy to see that this results in a configuration in which $2k + 3$ bisectors are incident on point p, $2k + 1$ bisectors are incident on each of the points q and r, 3 bisectors are on each of the k points in the center, and a single bisector is on each of the $3k + 1$ points in the negative cones. The resulting configuration has $4k + 4$ points and $6k + 3$ bisectors. The sum of the valences at points p, q, and r is

$$\nu_p + \nu_q + \nu_r = 6k + 5 = 3n - 1,$$

where $n = 2(k + 1)$ is even.

Figure 5 shows the modified central symmetric star bigraphs of orders 0, 1, 2, and 3.

These two constructions complete the proof of Theorem 5. ∎

It is worthy of note in closing that all of the results given in this paper, except for Theorem 1 and its corollary, are dependent only on the config-

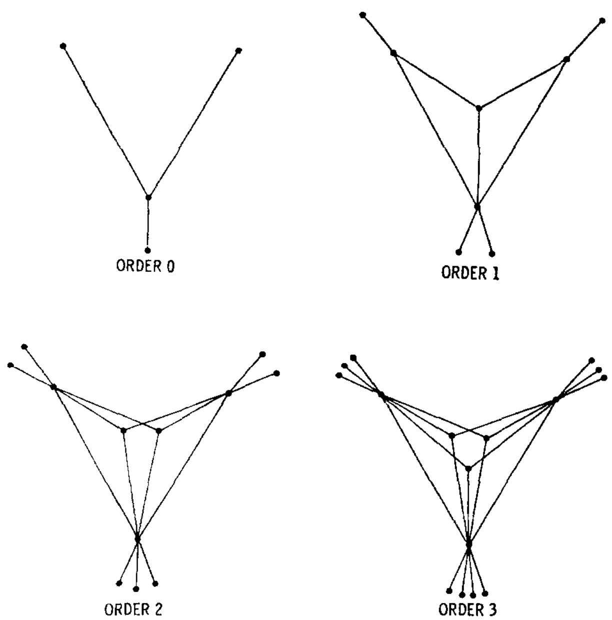
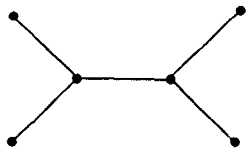


FIG. 5. Modified central symmetric star bigraphs.

uration of lines determined by the $2n$ points satisfying the negative cone condition of Theorem 1 and the odd valence condition of its corollary. Bigraphs have been proved to meet these conditions; however, there do exist configurations which satisfy these conditions but which cannot be bigraphs. The simplest example is the following configuration on six points:



Therefore the conclusions arrived at here concerning the sums of valences all apply to this more general class of graphs which include bigraphs as a special subclass.

REFERENCES

1. L. LOVÁSZ, On the number of halving lines, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **14** (1971), 107–108.